# Technical report: LB model with adjustable speed of sound

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In the most commonly implemented lattice Boltzmann models, the speed of sound $v_s^2$ is fixed: it is equal to the lattice constant $c_s^2 \equiv 1/3$. As it is shown in [Chopard e.a. (2002), Adv. Compl. Sys. 5, p.103-146], it is possible to adjust the speed of sound when the zero-particle population $f_0$ follows a formally different expression than the non-zero ones. The BGK model with adjustable speed of sound can be written as follows (make sure to understand that $v_s^2$ is a free parameter, whereas $c_s^2$ is a lattice constant):

$$
\begin{aligned}
f_i^{(0)} &= \rho t_i \frac{v_s^2}{c_s^2}\left(1 + \frac{c_i \cdot u}{v_s^2} + \frac{1}{2c_s^2 v_s^2}\left(c_i c_i - c_s^2 I\right) : uu\right) \quad \text{for} \quad i = 1 \cdots z \quad \textit{and} \\
f_0^{(0)} &= \rho\left(1 - \frac{v_s^2}{c_s^2}(1 - t_0) - \frac{t_0}{2c_s^2}\,|u|^2\right)
\end{aligned}
$$

The lattice constants $t_i$ and $c_s^2$ must verify the following constraints:

$$(a)\ \sum_i t_i = 1 \qquad (c)\ \sum_i t_i c_{i\alpha} c_{i\beta} = c_s^2 \delta_{\alpha\beta} \qquad (e)\ \sum_i t_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} = c_s^4\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$$

$$(b)\ \sum_i t_i c_{i\alpha} = 0 \qquad (d)\ \sum_i t_i c_{i\alpha} c_{i\beta} c_{i\gamma} = 0 \qquad (f)\ \sum_i t_i c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\delta} c_{i\epsilon} = 0.$$

From these, it is easy to verify that the equilibrium above has the expected moments:

$$
\begin{aligned}
\sum_{i=0}^{z} f_i^{(0)} &= \rho \\
\sum_{i=0}^{z} f_i^{(0)} c_i &= \rho u \\
\sum_{i=0}^{z} f_i^{(0)} c_i c_i &= v_s^2 \rho + \rho u u \\
\nabla \cdot \sum_{i=0}^{z} f_i^{(0)} c_i c_i c_i &= v_s^2\left(\nabla(\rho u) + (\nabla(\rho u))^T + \nabla \cdot (\rho u) I\right)
\end{aligned}
$$

From the multi-scale analysis, it is then clear that this model represents a weakly compressible fluid with adjustable speed of sound $v_s$. The parameter $v_s^2$ is constrainted to reside in the following range:

$$0 < v_s^2 < \frac{c_s^2}{1 - t_0}.$$